Linear Filters in StreamIt

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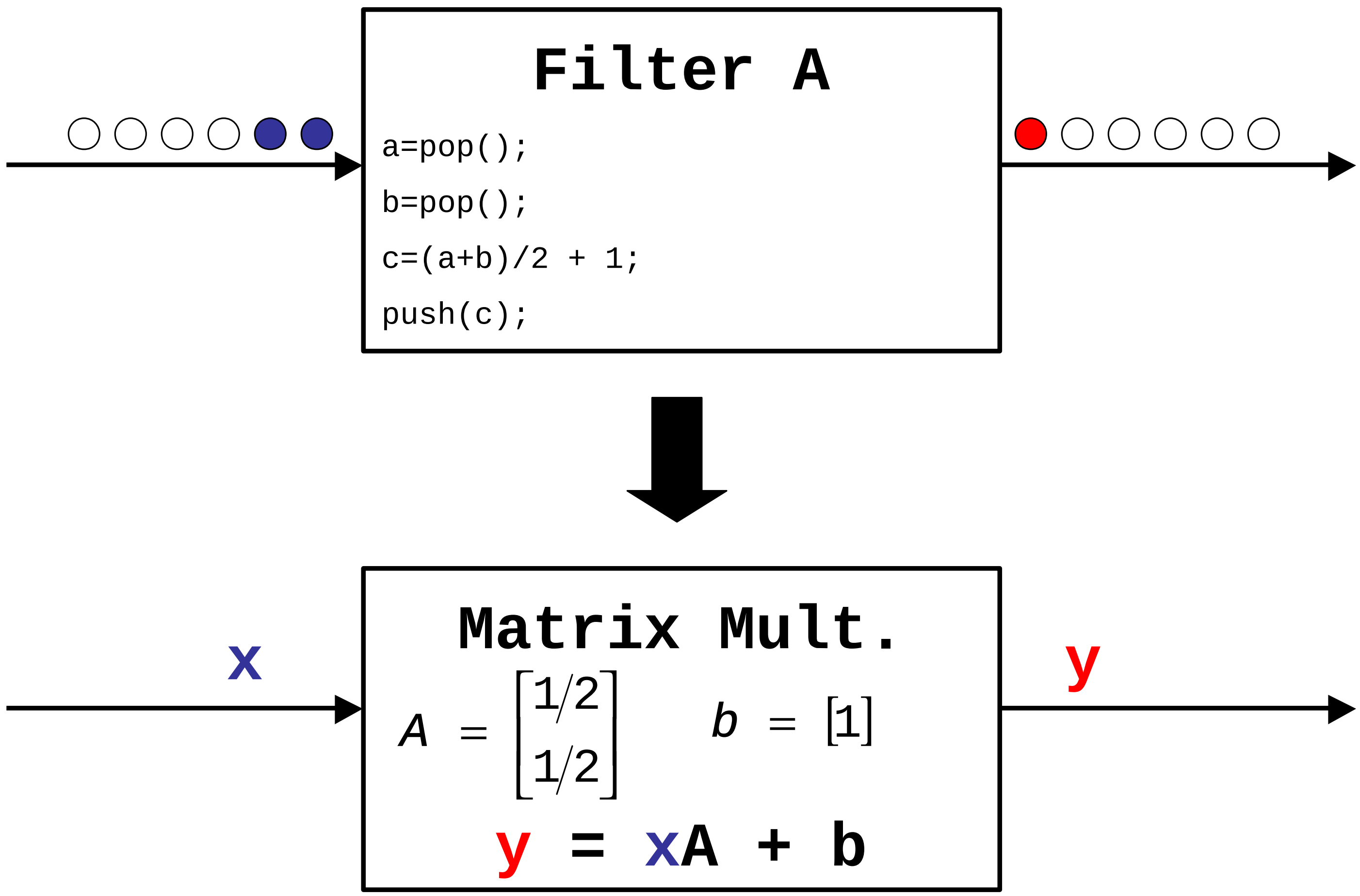
Outline

- Introduction
- Dataflow Analysis
- Hierarchal Matrix Combinations
- Performance Optimizations

Basic Idea



Filter A
a=pop();
b=pop();
c=(a+b)/2 + 1;
push(c);
x
Matrix Mult.
$A = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ $b = [1]$
y = xA + b
y

What is a Linear Filter?

- Generic filters calculate some outputs (possibly) based on their inputs.
- Linear filters: outputs (y_j) are weighted sums of the inputs (x_i) plus a constant.

$$y = \sum_{i \in [1,N]} w_i x_i + b$$

for b constant

w_i constant for all i

N is the number of inputs

$$y = w_1 x_1 + w_2 x_2 + (\ldots) + w_N x_N + b$$

Linearity and Matricies

- Matrix multiply is exactly weighted sum
- We treat inputs (x_i) and outputs (y_j) as vectors of values ($\mathbf{x}$, and $\mathbf{y}$ respectively)
- Filter is represented as a matrix of weights $\mathbf{A}$ and a vector of constants $\mathbf{b}$
- Therefore, filter represents the equation

  $\mathbf{y} = \mathbf{x}\mathbf{A} + \mathbf{b}$

Equation Example, y_1

$$
\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}
\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{bmatrix}
+ \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}
= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}
$$

$$y_1 = (x_1 a_{1,1} + x a_{2,1} + x_3 a_{3,1} + x_4 a_{4,1} + b_1)$$

Equation Example, y_2

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

$$y_2 = (x_1 a_{1,2} + x a_{2,2} + x_3 a_{3,2} + x_4 a_{4,2} + b_2)$$

Equation Example, y_3

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

$$y_3 = (x_1 a_{1,3} + x a_{2,3} + x_3 a_{3,3} + x_4 a_{4,3} + b_3)$$

Usefulness of Linearity

- Not all filters compute linear functions
  - `push(pop()*pop());`

- Many fundamental DSP filters do
  - DFT/FFT
  - DCT
  - Convolution/FIR
  - Matrix Multiply

Example: DFT Matrix

DFT: $X(m) \equiv \sum_{n=0}^{N-1} x(n) e^{-j2pnm/N}, m = 0,1,2,\ldots, N-1$

$$F_N = \begin{bmatrix} w_N^{0\bullet 0} & w_N^{0\bullet 1} & \ldots & w_N^{0\bullet (N-1)} \\ w_N^{1\bullet 0} & w_N^{1\bullet 1} & \ldots & w_N^{1\bullet (N-1)} \\ \ldots & \ldots & \ldots & \ldots \\ w_N^{(N-1)\bullet 0} & w_N^{(N-1)\bullet 1} & \ldots & w_N^{(N-1)\bullet (N-1)} \end{bmatrix}$$

row n

column m

$$w_N = e^{-j2p/N}$$

Example: IDFT Matrix

IDFT: $x(n) \equiv \frac{1}{N}\sum_{m=0}^{N-1} X(m) e^{j2pmn/N}, n = 0,1,2,\ldots, N-1$

$$F_N^{-1} = \begin{bmatrix} w_N^{0\bullet 0} & w_N^{0\bullet 1} & \cdots & w_N^{0\bullet(N-1)} \\ w_N^{1\bullet 0} & w_N^{1\bullet 1} & \cdots & w_N^{1\bullet(N-1)} \\ \cdots & \cdots & \cdots & \cdots \\ w_N^{(N-1)\bullet 0} & w_N^{(N-1)\bullet 1} & \cdots & w_N^{(N-1)\bullet(N-1)} \end{bmatrix}$$

row m

column n

$$w_N = \left(\frac{1}{N}\right) e^{j2p/N}$$

Usefullness, cont.

- Matrix representations
  - Are “embarrassingly parallel” [1]
  - Expose redundant computation
  - Let us take advantage of existing work in DSP field
  - Well understood mathematics

1: Thank you, Bill Thies

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Dataflow Analysis

- Basic idea: convert the general code of a filter's work function into an affine representation (eg $\mathbf{y}=\mathbf{x}\mathbf{A}+\mathbf{b}$)
  - The **A** matrix represents the linear combination of inputs used to calculate each output.
  - The vector **b** represents a constant offset that is added to the combination.

“Linear” Dataflow Analysis

- Much like standard constant prop.
- Goal: Have a vector of weights and a constant that represents the argument to each push statement which become a column in **A** and an entry in **b**.
- Keep mappings from variables to their linear forms (eg vector + constant).

"Linear" Dataflow Analysis

- Of course, we need the appropriate generating cases, eg
  - constants $\longrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + c$
  - pop/peek(x) $\longrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0$

"Linear" Dataflow Analysis

- Like const prop, confluence operator is set union.
- Need combination rules to handle things like multiplication and addition (vector add and constant scale)

Ridiculous Example

```
a=peek(2);
b=pop();
c=pop();
pop();
d=a+2b;
e=d+5;
```

Ridiculous Example

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$$\mathbf{a} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0$$

Ridiculous Example

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a=peek(2);
b=pop();
c=pop();
pop();
d=a+2b;
e=d+5;
```

$$\mathbf{a} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0$$

$$\mathbf{b} \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0$$

Ridiculous Example

```
a=peek(2);
b=pop();
c=pop();
pop();
d=a+2b;
e=d+5;
```

$$\mathbf{a} \longrightarrow \begin{bmatrix}1\\0\\0\end{bmatrix}+0$$

$$\mathbf{b} \longrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix}+0$$

$$\mathbf{c} \longrightarrow \begin{bmatrix}0\\1\\0\end{bmatrix}+0$$

Ridiculous Example

```
a=peek(2);
b=pop();
c=pop();
pop();
d=a+2b;
e=d+5;
```

$$\mathbf{a} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0$$

$$\mathbf{b} \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0$$

$$\mathbf{c} \longrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0$$

$$\mathbf{d} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 0$$

Ridiculous Example

```
a=peek(2);
b=pop();
c=pop();
pop();
d=a+2b;
e=d+5;
```

$$\mathbf{a} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0$$

$$\mathbf{b} \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0$$

$$\mathbf{c} \longrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0$$

$$\mathbf{d} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 0$$

$$\mathbf{e} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 5$$

Constructing matrix **A**

Filter A:

filter code

```
push(b);
push(a);
```

$$\mathbf{a} \longrightarrow \begin{bmatrix}1\\2\\3\end{bmatrix}+4$$

$$\mathbf{b} \longrightarrow \begin{bmatrix}5\\6\\7\end{bmatrix}+8$$

$$A=\begin{bmatrix}0 & 0\\0 & 0\\0 & 0\end{bmatrix} \quad b=\begin{bmatrix}0 & 0\end{bmatrix}$$

Constructing matrix **A**

Filter A:

filter code

```
push(b);
push(a);
```

$$\mathbf{a} \longrightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 4$$

$$\mathbf{b} \longrightarrow \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} + 8$$

$$A = \begin{bmatrix} 0 & 5 \\ 0 & 6 \\ 0 & 7 \end{bmatrix} \quad b = \begin{bmatrix} 0 & 8 \end{bmatrix}$$

Constructing matrix **A**

Filter A:

filter code

```
push(b);
push(a);
```

a $\longrightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 4$

b $\longrightarrow \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} + 8$

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \end{bmatrix} \quad b = \begin{bmatrix} 4 & 8 \end{bmatrix}$$

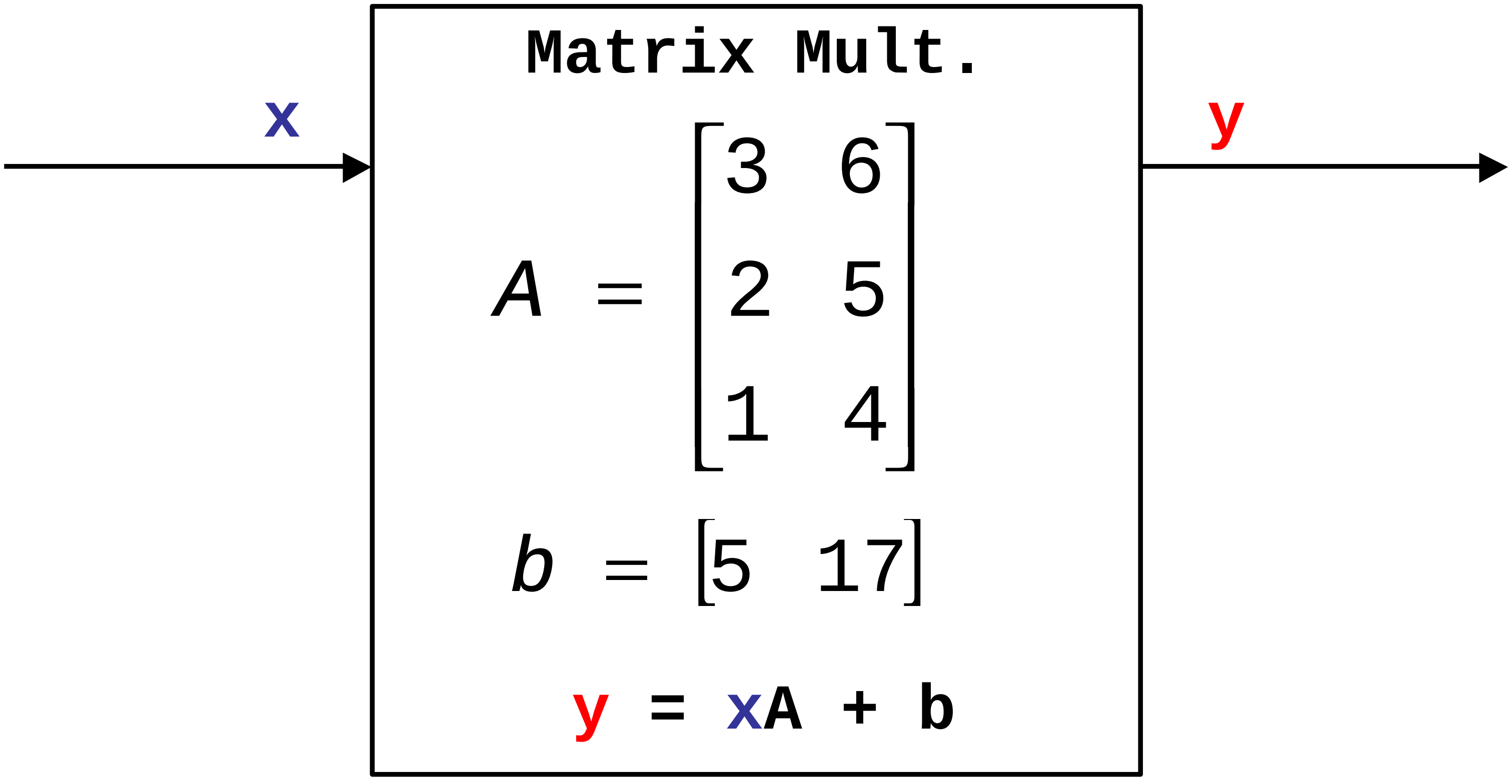
Big Picture

FIR Filter

```
weights1 = {1,2,3};
weights2 = {4,5,6};
float sum1 = 0; float sum2 = 0;
float mean1 = 5; mean2 = 17;
for (int i=0; i<3; i++) {
    sum1 += weights1[i]*peek(3-i-1);
    sum1 += weights1[i]*peek(3-i-1);
}
push(sum2 – mean2);
push(sum1 – mean1);
pop();
```

```
push = 2
pop  = 1
peek = 3
```

Big Picture, cont.



$$A = \begin{bmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 & 17 \end{bmatrix}$$

y = xA + b

```
size(x) = 3
size(y) = 2
```

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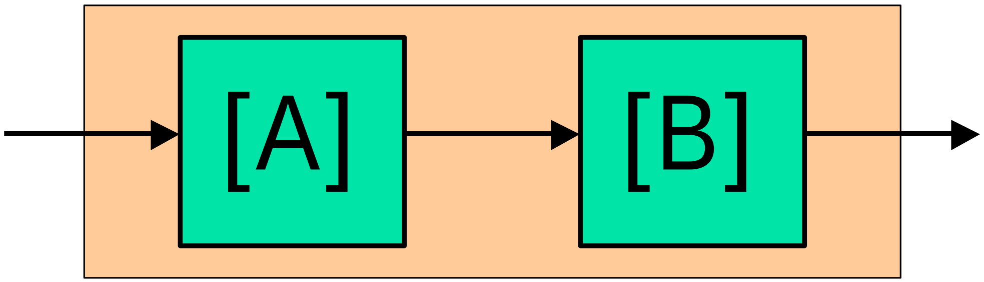
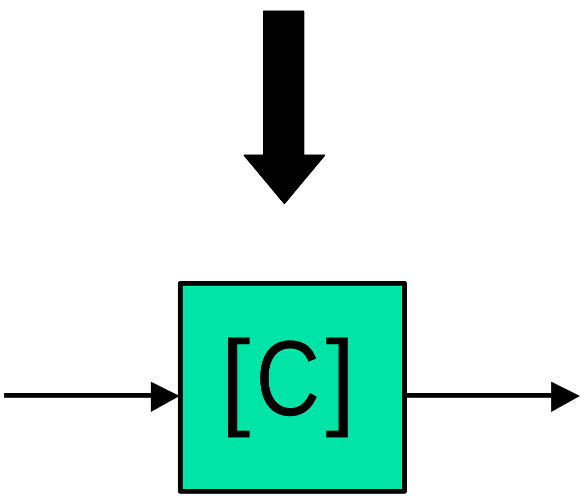
Combining Filters

- Basic idea: combine pipelines, splitjoins (and possibly feedback loops) of linear filters together
- End up with a single large matrix representation
- The details are tricky in the general case (eg I am still working on them)

Combining Pipelines



ye olde pipeline

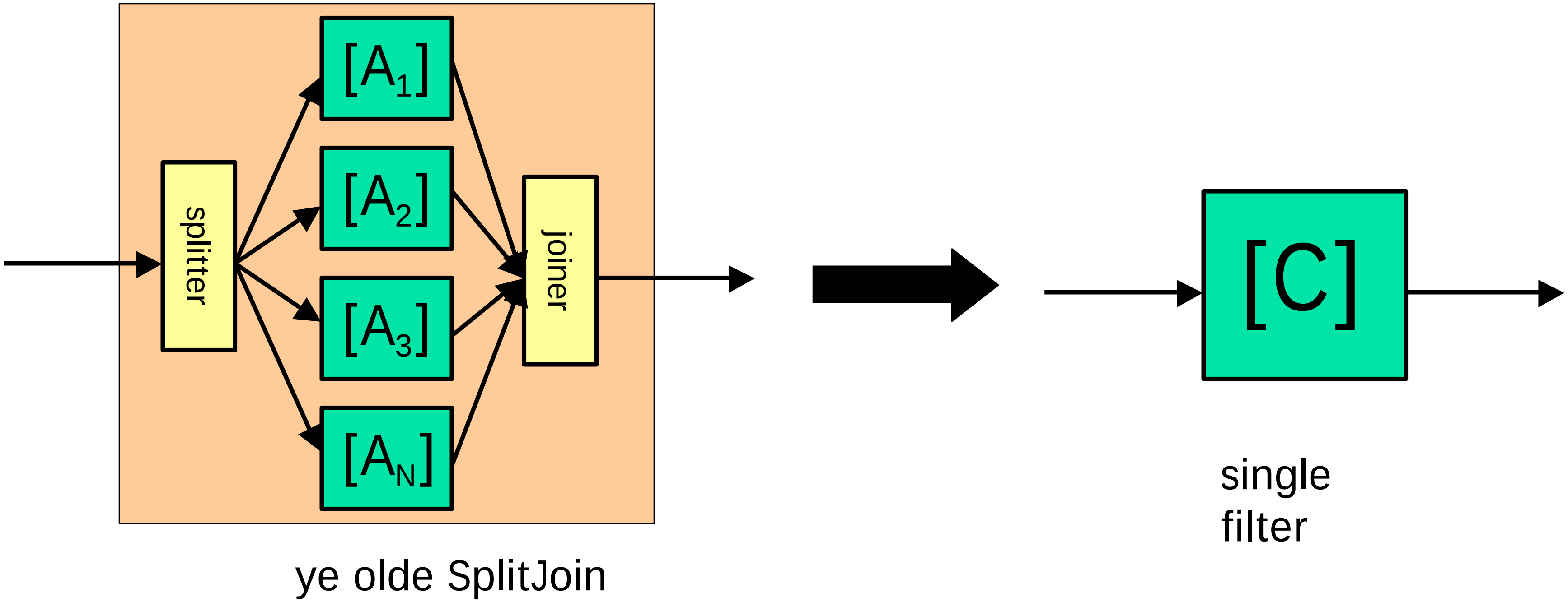


single filter

The matrix **C** is calculated as **A’ B’** where **A’** and **B’** have been appropriately scaled and duplicated to make the dimensions work out.

In the case where peek(B) $\neq$ pop(B), we might have to use two stage filters or duplicate some work to get the dimensions to work out.

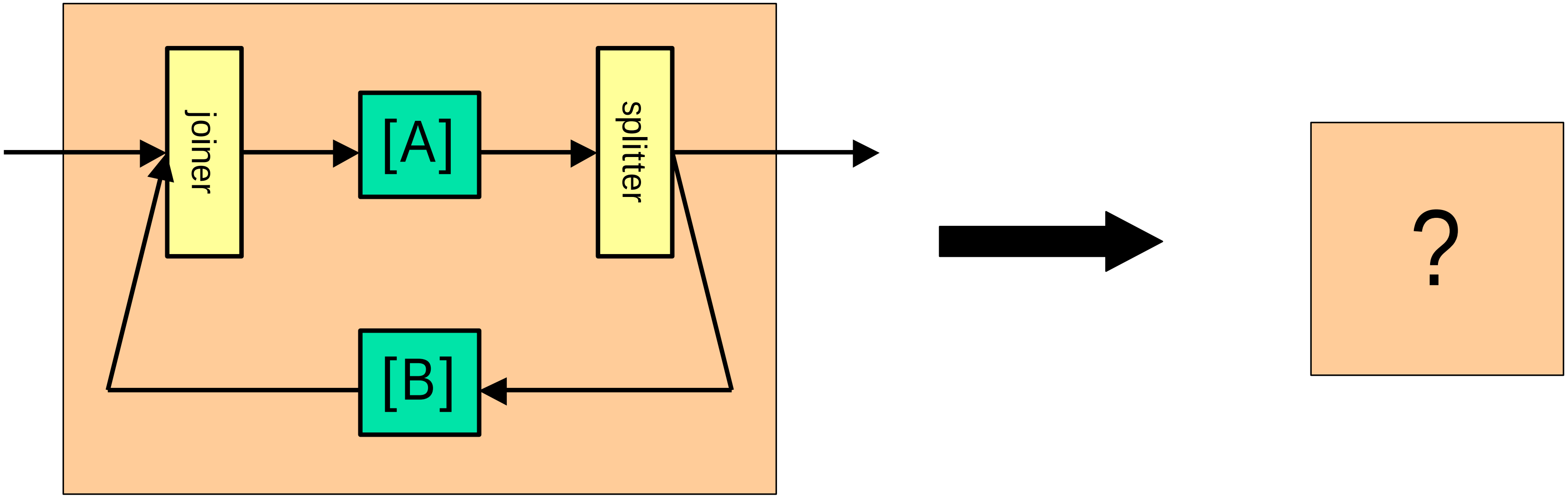
Combining Split Joins



A split join reorders data, so the columns of **C** are interleaved copies of the columns of A_1 through A_N.

Matching the rates of A_1 through A_N is a challenge that I am still working out.

Combining Feedback Loops



ye olde FeedbackLoop

It is unclear if we can do anything of use with a FeedbackLoop construct. Eigen values might give information about stability, but it is not clear if that is useful... more thought is needed.

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Performance Optimizations

- Take advantage of our compile time knowledge of the matrix coefficients.
  - eg don't waste computation on zeros
- Try and leverage existing DSP work on factoring matricies.
- Try to recognize parallel structures in our matrices.
- Use frequency analysis.

Factoring for Performance

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

16 multiplies

12 adds

$$BC = \begin{bmatrix} b_{1,1} & 0 & 0 & 0 \\ 0 & b_{2,2} & 0 & 0 \\ 0 & 0 & b_{3,3} & 0 \\ 0 & 0 & 0 & b_{4,4} \end{bmatrix} \begin{bmatrix} c_{1,1} & 0 & 0 & 0 \\ c_{2,1} & c_{2,2} & 0 & 0 \\ c_{3,1} & c_{3,2} & c_{3,3} & 0 \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{bmatrix}$$

14 multiplies

6 adds

SPL/SPIRAL

- Software package that will attempt to find a fast implementation signal processing algorithms described as matrices.
- It attempts to find a sparse factorization of an arbitrary matrix.
- It can automatically derive FFT (eg the Cooley-Turkey algorithm) from DFT definition.
- Claim that their performance is ≈ FFTW[1].

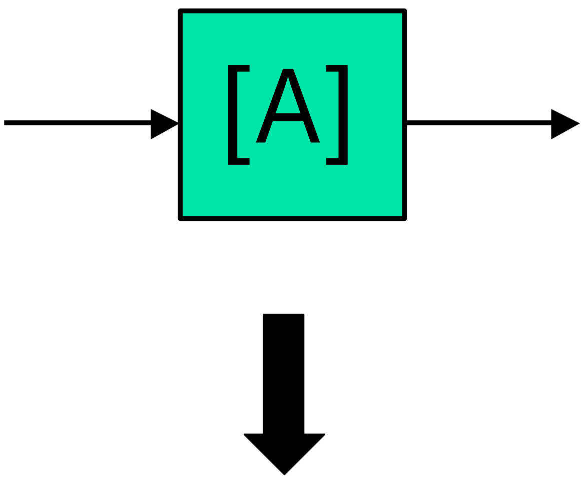
1. See http://www.fftw.org

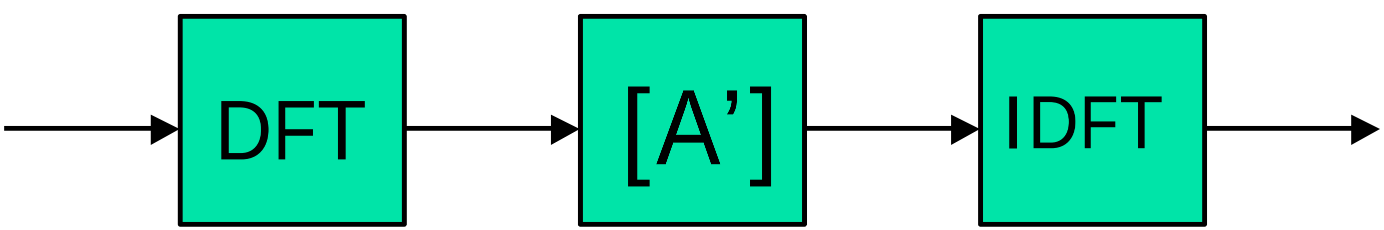
Recognize Parallel Structure

- We can go from SplitJoin to matrix.
- Perhaps we can recognize the reverse transformation.
- Also, implement blocked matrix multiply to keep parallel resources busy.

Frequency Analysis

- Instead of computing the matrix product straight up, possibly go to frequency domain.
- Rids us of offset vector (added to response at f=0).
- Might allow additional optimizations (because of possible symmetries exposed in frequency domain).





Work left to do

- Implementation of single filter analysis.
- Combining hierarchical constructs.
- Understand the math of automatic matrix factorizations (group theory).
- Analyze frequency analysis.
- Implement optimizations.
- Get results.

Questions for the Future

- Are there any other optimizations?
- Can we produce inverted matrices
  - programmer codes up transmitter and StreamIt automatically creates the receiver.[1]
- How many cycles of a “real” DSP application are spent computing linear functions?
- Can we combine the linear description of what happens inside a filter with the SARE representation of what is happening between them? (POPL paper)

1. Thank you, BT